INTRODUCTION

Aim of the Project
Establish a direct relationship between differential entropy and variance for symmetric unimodal distributions.

Motivation
- Estimate the continuous random variable X by querying the oracle.
- Similar to the communication with noiseless feedback.
- Minimize the mean squared error $E[|X - X'(Y^n)|^2]$.
- Q: What is the optimal querying strategy for this noisy case?
- Successive entropy minimization is often proposed as a way to progressively concentrate the posterior distribution.

Relationship Between Variance and Differential Entropy

$$\text{var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 p(x) dx,$$

$$h(p) = -\int_{-\infty}^{\infty} p(x) \log p(x) dx.$$

- No universal direct relation between these measures.
- The estimation counterpart to the Fano’s inequality [1] shows $\frac{1}{2} e^{2h(p)} \leq \text{var}(X)$.

In general, no upper bound on variance in terms of differential entropy.

FINITE VARIANCE RATIO

Necessity of Bounded Variance Ratio $r$
In Theorem 1, we assumed boundedness of the ratio $r$ between the maximum and minimum variances of the mixture components. We show by a counterexample that this assumption is necessary.

Counterexample
A fixed $\epsilon_2 > 0$

$\epsilon_1 \rightarrow \infty$

$\epsilon_2 \rightarrow 0$

- Consider $p(x) = \sum_{i=1}^{n} \alpha_i p_i(x)$ for $\alpha_i \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$.

- The variance is proportional to the arithmetic mean of $\alpha_i^2$.

- When $\epsilon_2 \rightarrow \infty$, the entropy power is proportional to the geometric mean, i.e., $e^{2h(p)} \propto \sqrt[n]{\prod_{i=1}^{n} \alpha_i^{2\epsilon_2}}$.

- If $\epsilon_2 \rightarrow \infty$, variance increases much faster than $e^{2h(p)}$.

- Boundness of $r$ is necessary for existence of an upper bound on variance in form of $e^{2h(p)}$.

LIPSCHITZ CONTINUOUS DENSITIES

Lipschitz Continuous Symmetric Unimodal Density with Bounded Support
- Suppose that a symmetric unimodal density $p$ with support $[m-s, m+s]$ satisfies the Lipschitz condition with constant $c_l > 0$, i.e.,

$$|p(x + y) - p(x)| \leq c_l |y|$$

for any $x, y \in [m - s, m + s]$.

- The Lipschitz constant $c_l$ and the support size $s$ determines the tightness of the upper bound.

Theorem 2 (Chung, Sadler, and Hero 2015). For any Lipschitz continuous symmetric unimodal density $p(x)$ with bounded support $[m - s, m + s]$, we have

$$\frac{e^{2h(p)}}{2c_l} \leq \text{var}(X) \leq \frac{c_l s^2 e^{2\epsilon_2}}{24} e^{2h(p)}.$$

Sketch of Proof: Approximation of Symmetric Unimodal Distribution
- Construct a linear mixture density $p_n(x) = \sum_{i=1}^{n} \alpha_i \cdot \text{unif}(-th_i, th_i)$ that approximates $p(x)$ such that

$$e^{2h(p_n)} \leq e^{2h(p)} (1 + c_l n^{-1} \log n);$$

$$|\text{var}(p) - \text{var}(p_n)| \leq c_l n^{-1}$$

for some constants $c_l, c_2 > 0$.

- Prove an upper bound on variance of $p_n(x)$ in terms of $e^{2h(p_n)}$:

$$\text{var}(p) \leq \frac{c_l s^2 e^{2\epsilon_2}}{24} e^{2h(p_n)} (1 + c_l n^{-1}).$$

- By combining these results and letting $n \rightarrow \infty$,

$$\text{var}(p) \leq \frac{c_l s^2 e^{2\epsilon_2}}{24} e^{2h(p)}.$$

SUMMARY

Conditional Entropy as a Surrogate for MSE
- The variance of the general class of symmetric unimodal distribution can be bounded below and above by a constant scaling of its entropy $e^{2h(p)}$.

$$\frac{e^{2h(p)}}{2c_l} \leq \text{var}(X) \leq e^{2h(p)}.$$

- Provide justification to successive entropy minimization in Bayesian sequential query design when the posterior is symmetric and unimodal.

- Future work: can we remove symmetry condition? extension to multi-dimensional case?

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Reference:

MAIN CONTRIBUTIONS

We establish the complementary result that such an upper bound on variance in terms of the entropy power, i.e.,

$$\frac{e^{2h(p)}}{2c_l} \leq \text{var}(X) \leq c \cdot e^{2h(p)}, \quad c > 0$$

extends to the general class of symmetric unimodal distributions.