Optimal measurement matrix design for structured signals in clutter and noise

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Problem setup

$$Y = \Phi(X + C) + W$$

. The signal X and clutter C $\in \mathbb{R}^N$ and observation noise W $\in \mathbb{R}^K$ are assumed to be drawn from Gaussian distributions $\mathcal{N}(0, \Sigma_X)$ $\mathcal{N}(0, \Sigma_C)$, $\mathcal{N}(0, \sigma_n^2 I)$, respectively with low-rank covariance matrices, where $rank(\Sigma_C) = r_c$, and $rank(\Sigma_X) = r_X$

. We consider the problem of sensing matrix design to minimize Mean Square Error (MSE) under a power constraint on the sensing matrix.

$$\Phi^* = \arg\min_{\Phi} Tr\left[\Sigma_X\left(I + \frac{\Sigma_X^{1/2}\Phi^T}{\sigma_n^2}\left(\frac{\Phi\Sigma_C\Phi^T}{\sigma_n^2} + I\right)^{-1}\Phi\Sigma_X^{1/2}\right)^{-1}\right] \quad (1)$$

$$Tr\left(\Phi^*\Phi^{*T}\right) \le P$$

Phase transition analysis

Since the signal constitutes a low-rank subspace, we consider compressive measurement mechanism with number of measurements K much less than the dimensionality of the original space N.

. Renna, Calderbank et al. [1] have shown that for the case without clutter the number of measurements required should equal the rank of the signal subspace for phase transition to occur in the noiseless case.

. We present conditions when phase transition occurs in the presence of clutter as noise variance goes to 0. The error floor in the presence of clutter goes to 0 if

- number of measurements $K \ge r_X$.
- $r_X + r_C \le N$
- each $\forall v \in \bar{V}_x^R : v \not\subset \bar{V}_c^R$

where $\bar{V}_x^R$ are the Principal components of range of projected signal, while $\bar{V}_c^R$ are the Principal components of projected clutter,

Measurement matrix design

. [1] also gives a mechanism for designing matrices using a simple water-filling technique by exploiting the schur concavity of the objective function.

. In our case, the objective function can be expressed as

$$m(\Phi; \sigma_n^2) = Tr\left[\Lambda_X R(\Phi)\right]$$

$$R(\Phi) = \left(I + \frac{\Lambda_X^{1/2} V_X^T \Phi^T}{\sigma_n^2}\left(\frac{\Phi\Sigma_C\Phi^T}{\sigma_n^2} + I\right)^{-1}\Phi V_X \Lambda_X^{1/2}\right)^{-1}$$

. The MSE is schur concave in the MSE matrix and the minimum can be attained if it is diagonalized as shown by Palomar et. al. [3].

. In order to get a more tractable solution, we transform the problem by applying a whitening filter as suggested in Kalyani et. al. [2] to transform the problem of the form in [1].

Whitening transformation

. The whitening filter used is of the form $C_\Phi = \left(\frac{\Phi\Sigma_C\Phi^T}{\sigma_n^2} + I\right)^{-1/2}$.

. The transformed observation is $Z = AX + N$.

. The optimal measurement matrix for the modified problem is of the form

$$A^* = \begin{bmatrix} \sqrt{\Lambda_A(1)} & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & \sqrt{\Lambda_A(K)} & \cdots & 0 \end{bmatrix} V_X^T$$

. Given A^*, we can find a corresponding $\Phi^* = \left(I - \frac{A^*\Sigma_C A^{*T}}{\sigma_n^2}\right)^{-1/2} A$.

Modified Problem

. The optimization problem for the transformed variable is

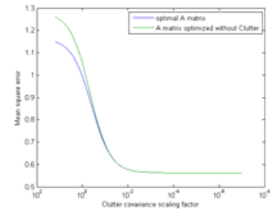
$$\min_{\Lambda_A} \sum_{i=1}^{K} \frac{\lambda_A(i)}{1 + \frac{\lambda_X(i)\lambda_A(i)}{\sigma_n^2}} \text{ subject to } Tr\left[\left(I - [\bar{\Lambda}_A \; 0]\frac{V_X^T\Sigma_C V_X}{\sigma_n^2}\begin{bmatrix}\bar{\Lambda}_A \\ 0^T\end{bmatrix}\right)^{-1}\bar{\Lambda}_A^2\right] \le P$$

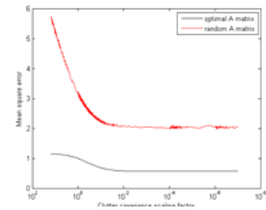
Jointly diagonalizable signal and clutter

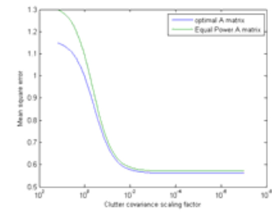
. For the case when signal and clutter share the eigen-space, the non-linear constraint simplifies significantly. The solution to that problem is akin to the water-filling solution and is of the form,

$$\lambda_A(i) = \frac{\left(1 - \frac{\sqrt{\beta}\sigma}{\lambda_X(i)}\right)_+}{\frac{\sqrt{\beta}}{\sigma_n} + \frac{\lambda_C(i)}{\sigma_n^2}}, \text{ and } \beta > 0 \text{ such that } \sum_{i=1}^{K} \frac{\lambda_A(i)}{1 - \frac{\lambda_A(i)\lambda_C(i)}{\sigma_n^2}} \le P .$$

Example for Jointly diagonalizable case







Mean square error of optimal matrix with white noise

Future Work

- Consider more general signal distributions for measurement matrix design.
- Extend measurement matrix design for multi-class classification.

References

[1] "Reconstruction of Signals Drawn From a Gaussian Mixture Via Noisy Compressive Measurements," Renna, F.; Calderbank, R.; Carin, L.; Rodrigues, M.R.D., *Signal Processing, IEEE Transactions on* , vol.62, no.9, pp.2265,2277, May1, 2014.

[2] "Target detection performance bounds in compressive imaging", Kalyani Krishnamurthy, Rebecca Willett, and Maxim Raginsky. EURASIP Journal on Advances in Signal Processing, 2012.

[3] "Joint Tx-Rx beamforming design for multicarrier MIMO channels: a unified framework for convex optimization". Palomar, D.P, Cioffi, J.M. ; Lagunas, Miguel Angel; Signal Processing, IEEE Transactions on (Volume:51 , Issue: 9), pp. 2381 – 2401, August 26, 2003.